AN INVESTIGATION OF THE QUASI-INVARIANTS OF THE STATIC-GEOMETRIC ANALOGY FOR THIN ELASTIC SHELLS

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The equations of the theory of thin elastic shells possess a characteristic symmetry which has enabled an intrinsic analogy to be established between the elastic constants and the static and geometric elements which appear in the formulation of the theory. This is the so-called static-geometric analogy which was formulated by Goldenveizer for isotropic shells [1]. The present authors later generalized the static-geometric analogy to cover the cases of orthotropic [2] and anisotropic shells [3].

The static-geometric analogy enables us to write down the equations of the theory of shells in complex form. The results obtained by Novozhilov in this field are well known [4].

In the case of cylindrical isotropic shells of arbitrary section, Novozhilov reduced the set of equations of the theory of shells to a single equation [4].

An analogous result was obtained for a thin spherical shell [5] by Goldenveizer, who derived a set of equations in complex displacements for an isotropic shell.

In [7] the static-geometric analogy is used to reduce the general equations for thin isotropic shells to a single complex equation of the fourth order.

With the aim of applying these methods to the case of orthotropic shells, the authors found that the factor

$$\frac{2h^2\sqrt{E_\alpha E_\beta}}{\sqrt{3(1-\mu_\alpha\mu_\beta)}}$$

enables the sets of equations of equilibrium and continuity to be combined to form a single complex set, and Hooke's equations to be reduced to a set of three linear non-differential equations in complex forces [8].

It is well known that the static-geometric analogy enables each relation found to be duplicated. These results then suggest the formulation of a single theory of thin shells in the complex domain which would reduce the number of unknowns and the order of the equations by half.

1. Quasi-invariants. The static-geometric analogy shows that the forces, moments, stress functions, displacements and strain components which appear in the homogeneous equations of the theory of thin shells can be divided into two groups (see appendix): so that each element of the first group containing forces, moments and stress functions corresponds to an element of the second group containing displacements and strains.

A fraction comprising analogous elements, in which the numerator belongs to the first group and the denominator to the second, has the dimensions of a force.

An expression relating the elements of one group only, will be considered as belonging to this group, so that the equilibrium equation in terms of forces, for example, might refer to the first group, and the equation of continuity of deformation in terms of strains to the second, etc. There are, in addition, equations which do not belong to the first or the second group. Hooke's law, for example, would come within this category.

Let us suppose that e is an element occurring in the first group (a force, a stress function or any homogeneous relation between the elements of the first group), and that e^* is its corresponding element in the second group.

We shall define a complex element as an element of the form

$$S_e = e + i\xi(e)\, e^* \tag{1.1}$$

In the static-geometric analogy a quantity which corresponds to itself can be called an invariant. For example, $1/3\, h^2(1 - \mu^2)$ is an invariant. The concept of an invariant imposes considerable restrictions, and having in mind the investigation of a set of homogeneous equations of the theory of shells we shall introduce the following more general concept.

We shall define a quasi-invariant as a complex element of the form

(1.1) which in the static-geometric analogy corresponds to the same element multiplied by a constant coefficient. We shall find the condition for which the complex element S_e is a quasi-invariant.

Let us suppose that in the static-geometric analogy $\xi^*(e)$ is a quantity corresponding to $\xi(e)$, and $S_e{}^*$ is a quantity corresponding to S_e. For (1.1), applying the static-geometric analogy, we obtain

$$S_e^* = e^* + i\xi^*(e)\,e \tag{1.2}$$

The condition of quasi-invariance for the element S_e can be written as

$$S_e = KS_e^* \tag{1.3}$$

or alternatively

$$e + i\xi(e)\,e^* = K\,[e^* + i\xi^*(e)\,e]$$

From this, after identifying the coefficients of the elements e and e^*, we have

$$1 = Ki\xi^*(e), \qquad i\xi(e) = K \tag{1.4}$$

and consequently, eliminating K, we obtain

$$\xi^*(e) = -\frac{1}{\xi(e)} \tag{1.5}$$

Since the complex element S_e must be homogeneous dimensionally, according to the remarks made at the beginning of the present section, it follows that $\xi(e)$ has the dimensions of a force. Thus

$$|\xi(e)| = |F| \tag{1.6}$$

A more general expression for $\xi(e)$, composed of all the constants appearing in the static-geometric analogy [3], is of the form

$$\xi(e) = F_1{}^m F_2{}^{m'} D_1{}^p D_2{}^{p'} \left(\frac{A_{12}}{A_{22}}\right)^q \left(\frac{A_{21}}{A_{22}}\right)^{q'} \left(2\,\frac{A_{13}}{A_{22}}\right)^r \left(2\,\frac{A_{23}}{A_{22}}\right)^{r'} \left(2\,\frac{A_{31}}{A_{22}}\right)^s \left(2\,\frac{A_{32}}{A_{22}}\right)^{s'} \times$$
$$\times \left(4\,\frac{A_{33}}{A_{22}}\right)^t \left(\frac{a_{12}}{a_{11}}\right)^u \left(\frac{a_{21}}{a_{11}}\right)^{u'} \left(-\frac{a_{13}}{a_{11}}\right)^v \left(-\frac{a_{23}}{a_{11}}\right)^{v'} \left(-\frac{a_{31}}{a_{11}}\right)^z \left(-\frac{a_{32}}{a_{11}}\right)^{z'} \left(\frac{a_{33}}{a_{11}}\right)^w \tag{1.7}$$

Expressions (1.5), (1.6) and (1.7) show that a further requirement must be satisfied, that $\xi(e)$ is independent of the chosen element e. In this case it is necessary that

$$m = m',\ p = p',\ q = q',\ r = r',\ s = s',\ u = u',\ v = v',\ z = z'$$

From this

$$\xi = (F_1F_2)^m (D_1D_2)^p \left(\frac{A_{12}A_{21}}{A_{22}^2}\right)^q \left(4\frac{A_{13}A_{23}}{A_{22}^2}\right)^r \left(4\frac{A_{31}A_{32}}{A_{22}^2}\right)^s \left(4\frac{A_{33}}{A_{22}}\right)^t \times \times \left(\frac{a_{12}a_{21}}{a_{11}^2}\right)^u \left(\frac{a_{13}a_{23}}{a_{11}^2}\right)^v \left(\frac{a_{31}a_{32}}{a_{11}^2}\right)^z \left(\frac{a_{33}}{a_{11}}\right)^w \tag{1.8}$$

In future we shall write ξ for $\xi(e)$.

We shall apply condition (1.6); noting that

$$[F_1] = [F_2] = [F][L]^{-1}, \quad [D_1] = [D_2] = [F]^{-1}[L]^{-1}$$ ($[F]$ is the dimension of a force, $[L]$ of a length)

and that the remaining parentheses in Expressions (1.8) are dimensionless quantities, we have

$$m = \frac{1}{4}, \quad p = -\frac{1}{4}$$

the other indices remaining indeterminate for the time being.

If we apply condition (1.5), on the basis of the relations derived in [3], we have

$$q = -u, \quad r = -v, \quad s = -z, \quad t = -w$$

so that

$$\xi = \left(\frac{F_1F_2}{D_1D_2}\right)^{1/4} \left(\frac{A_{12}A_{21}}{a_{12}a_{21}}\frac{a_{11}^2}{A_{22}^2}\right)^q \left(4\frac{A_{13}A_{23}}{a_{13}a_{23}}\frac{a_{11}^2}{A_{22}^2}\right)^r \left(4\frac{A_{31}A_{32}}{a_{31}a_{32}}\frac{a_{11}^2}{A_{22}^2}\right)^s \left(4\frac{A_{33}}{a_{33}}\frac{a_{11}}{A_{22}}\right)^t \tag{1.9}$$

It will be noted that q, r, s, t remain arbitrary.

Note. If the coefficient ξ is determined such that

$$S_e = e + i\xi e$$

is a quasi-invariant, then

$$S_e' = e + i\xi\lambda e^*$$

is also a quasi-invariant if λ is dimensionless, and it satisfies in the static-geometric analogy the condition

$$\lambda \leftarrow\rightarrow \frac{1}{\lambda} \tag{1.10}$$

The proof of this is obvious.

In this connection we can discard in Expression (1.9) the indeterminate factors, which, as can easily be shown, satisfy condition (1.10). Consequently, we can take

$$\xi = \left(\frac{F_1F_2}{D_1D_2}\right)^{1/4} \tag{1.11}$$

Making use of the notation described in the appendix, we have for the general case

$$\xi = 2h^2 \sqrt[4]{\frac{1}{9}\frac{A_{11}A_{22}}{a_{11}a_{22}}} \tag{1.12}$$

or, if we make use of the engineering constants,

$$\xi = 2h^2 \sqrt{\frac{E_\alpha E_\beta}{3\Delta_1}} \sqrt[4]{(1-\eta_\alpha\nu_\alpha)(1-\eta_\beta\nu_\beta)} \tag{1.13}$$

where

$$\Delta_1 = \begin{vmatrix} 1 & -\mu_\alpha & \eta_\alpha \\ -\mu_\beta & 1 & \eta_\beta \\ \nu_\alpha & \nu_\beta & 1 \end{vmatrix} \tag{1.14}$$

In particular, we have from this that for isotropic and orthotropic shells, respectively,

$$\xi = \frac{2h^2E}{\sqrt{3(1-\mu^2)}}, \qquad \xi = \frac{2h^2\sqrt{E_\alpha E_\beta}}{\sqrt{3(1-\mu_\alpha\mu_\beta)}}$$

2. Formulation of the equations of the theory of thin elastic homogeneous shells in complex quantities. It follows from the above that groups of relations which correspond to each other in the static-geometric analogy can be combined to form quasi-invariant complex systems, the new functions introduced being quasi-invariants.

For example, the sets of equations of equilibrium in terms of forces and the equations of continuity of strain can be combined in this way to form a single set, and the new unknowns are complex quantities. Thus*

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* For simplicity, an orthogonal system of coordinates has been used, but the results remain valid in any other system.

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha}(B\boldsymbol{T}_1)+\frac{\partial A}{\partial\beta}\boldsymbol{S}_1-\frac{\partial}{\partial\beta}(A\boldsymbol{S}_2)-\frac{\partial B}{\partial\alpha}\boldsymbol{T}_2-AB\left(\frac{\boldsymbol{N}_1}{R_1'}-\frac{\boldsymbol{N}_2}{R_{12}}\right)+ABX=0\\
&\frac{\partial}{\partial\alpha}(B\boldsymbol{S}_1)-\frac{\partial A}{\partial\beta}\boldsymbol{T}_1+\frac{\partial}{\partial\beta}(A\boldsymbol{T}_2)-\frac{\partial B}{\partial\alpha}\boldsymbol{S}_2-AB\left(\frac{\boldsymbol{N}_2}{R_2'}-\frac{\boldsymbol{N}_1}{R_{12}}\right)+ABY=0\\
&AB\left(\frac{\boldsymbol{T}_1}{R_1'}+\frac{\boldsymbol{T}_2}{R_2'}+\frac{\boldsymbol{S}_2-\boldsymbol{S}_1}{R_{12}}\right)+\frac{\partial}{\partial\alpha}(B\boldsymbol{N}_1)+\frac{\partial}{\partial\beta}(A\boldsymbol{N}_2)+ABZ=0\\
&\frac{\partial}{\partial\alpha}(B\boldsymbol{H}_1)+\frac{\partial A}{\partial\beta}\boldsymbol{G}_1-\frac{\partial}{\partial\beta}(A\boldsymbol{G}_2)-\frac{\partial B}{\partial\alpha}\boldsymbol{H}_2+AB\boldsymbol{N}_2=0\\
&\frac{\partial}{\partial\alpha}(B\boldsymbol{G}_1)-\frac{\partial A}{\partial\beta}\boldsymbol{H}_1+\frac{\partial}{\partial\beta}(A\boldsymbol{H}_2)-\frac{\partial B}{\partial\alpha}\boldsymbol{G}_2-AB\boldsymbol{N}_1=0\\
&\boldsymbol{S}_1+\boldsymbol{S}_2+\frac{\boldsymbol{H}_1}{R_1'}+\frac{\boldsymbol{H}_2}{R_2'}+\frac{\boldsymbol{G}_2-\boldsymbol{G}_1}{R_{12}}=0
\end{aligned}
\qquad (2.1)
$$

Here

$$
\begin{aligned}
&\boldsymbol{T}_1=T_1+i\xi\varkappa_2, &&\boldsymbol{S}_1=S_1+i\xi\tau^{(2)}, &&\boldsymbol{N}_1=N_1-i\xi\zeta_2\\
&\boldsymbol{T}_2=T_2+i\xi\varkappa_1, &&\boldsymbol{S}_2=S_2+i\xi\tau^{(1)}, &&\boldsymbol{N}_2=N_2+i\xi\zeta_1\\
&\boldsymbol{G}_1=G_1+i\xi\varepsilon_2, &&\boldsymbol{H}_1=H_1-i\xi\omega^{(2)} &&\\
&\boldsymbol{G}_2=G_2+i\xi\varepsilon_1, &&\boldsymbol{H}_2=H_2-i\xi\omega^{(1)} &&
\end{aligned}
\qquad (2.2)
$$

It should be noted that to the set (2.1) must be added a further equation

$$\boldsymbol{H}_1+\boldsymbol{H}_2=0 \qquad (2.3)$$

The set of equations (2.1) is analogous in form to the set of equilibrium equations in the isotropic case.

The relations between the forces and stress functions on the one hand, and those between strains and displacements on the other, can also be combined to form a single set of relations between the complex forces and complex stress functions in the following manner:

$$
\begin{aligned}
\boldsymbol{T}_1&=\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial \boldsymbol{c}}{\partial\beta}+\frac{\boldsymbol{b}}{R_2'}-\frac{\boldsymbol{a}}{R_{12}}\right)+\frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{1}{A}\frac{\partial \boldsymbol{c}}{\partial\alpha}+\frac{\boldsymbol{a}}{R_1'}-\frac{\boldsymbol{b}}{R_{12}}\right)-\frac{\boldsymbol{n}}{R_{12}}\\
\boldsymbol{T}_2&=\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial \boldsymbol{c}}{\partial\alpha}+\frac{\boldsymbol{a}}{R_1'}-\frac{\boldsymbol{b}}{R_{12}}\right)+\frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{B}\frac{\partial \boldsymbol{c}}{\partial\beta}+\frac{\boldsymbol{b}}{R_2}-\frac{\boldsymbol{a}}{R_{12}}\right)+\frac{\boldsymbol{n}}{R_{12}}\\
\boldsymbol{S}_1&=-\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{A}\frac{\partial \boldsymbol{c}}{\partial\alpha}+\frac{\boldsymbol{a}}{R_1'}-\frac{\boldsymbol{b}}{R_{12}}\right)+\frac{1}{AB}\frac{\partial B}{\partial\alpha}\left(\frac{1}{B}\frac{\partial \boldsymbol{c}}{\partial\beta}+\frac{\boldsymbol{b}}{R_2'}-\frac{\boldsymbol{a}}{R'_2}\right)+\frac{\boldsymbol{n}}{R_2}\\
\boldsymbol{S}_2&=\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\frac{\partial \boldsymbol{c}}{\partial\beta}+\frac{\boldsymbol{b}}{R_2'}-\frac{\boldsymbol{a}}{R_{12}}\right)-\frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{A}\frac{\partial \boldsymbol{c}}{\partial\alpha}+\frac{\boldsymbol{a}}{R_1'}-\frac{\boldsymbol{b}}{R_{12}}\right)+\frac{\boldsymbol{n}}{R_1'}\\
\boldsymbol{N}_1&=-\frac{1}{B}\frac{\partial \boldsymbol{n}}{\partial\beta}-\frac{1}{R_2'}\left(\frac{1}{A}\frac{\partial \boldsymbol{c}}{\partial\alpha}+\frac{\boldsymbol{a}}{R_1'}-\frac{\boldsymbol{b}}{R_{12}}\right)-\frac{1}{R_{12}}\left(\frac{1}{B}\frac{\partial \boldsymbol{c}}{\partial\beta}+\frac{\boldsymbol{b}}{R_2'}-\frac{\boldsymbol{a}}{R_{12}}\right)\\
\boldsymbol{N}_2&=\frac{1}{A}\frac{\partial \boldsymbol{n}}{\partial\alpha}-\frac{1}{R_1'}\left(\frac{1}{B}\frac{\partial \boldsymbol{c}}{\partial\beta}+\frac{\boldsymbol{b}}{R_2'}-\frac{\boldsymbol{a}}{R_{12}}\right)-\frac{1}{R_{12}}\left(\frac{1}{A}\frac{\partial \boldsymbol{c}}{\partial\alpha}+\frac{\boldsymbol{a}}{R_1'}-\frac{\boldsymbol{b}}{R_{12}}\right)
\end{aligned}
\qquad (2.4)
$$

$$\boldsymbol{G}_1 = \frac{1}{B}\frac{\partial \boldsymbol{b}}{\partial \beta} + \frac{1}{AB}\frac{\partial B}{\partial \alpha}\boldsymbol{a} - \frac{\boldsymbol{c}}{R_2'}, \qquad \boldsymbol{H}_1 = \frac{1}{B}\frac{\partial \boldsymbol{a}}{\partial \beta} - \frac{1}{AB}\frac{\partial B}{\partial \alpha}\boldsymbol{b} + \frac{\boldsymbol{c}}{R_{12}} - \boldsymbol{n}$$

$$\boldsymbol{G}_2 = \frac{1}{A}\frac{\partial \boldsymbol{a}}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\boldsymbol{b} - \frac{\boldsymbol{c}}{R_1'} \qquad \boldsymbol{H}_2 = -\frac{1}{A}\frac{\partial \boldsymbol{b}}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\boldsymbol{a} - \frac{\boldsymbol{c}}{R_{12}} - \boldsymbol{n}$$

Here

$$\boldsymbol{a} = a + i\xi u, \quad \boldsymbol{b} = b + i\xi v, \quad \boldsymbol{c} = c + i\xi w \tag{2.5}$$

and $\boldsymbol{n}$ is denoted as

$$\boldsymbol{n} = \frac{1}{2AB}\left[\frac{\partial}{\partial \beta}(A\boldsymbol{a}) - \frac{\partial}{\partial \alpha}(B\boldsymbol{b})\right] \tag{2.6}$$

The functions $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ satisfy the condition of the homogeneous complex system (2.1), which is the reason why they are called complex stress functions. In view of (2.6) the complementary equation (2.3) is satisfied identically.

We shall consider now the case of equations containing heterogeneous quantities.

Two analogous equations can be combined to form a quasi-invariant complex in the form of a single equation expressed in quasi-invariant unknowns.

Suppose that

$$\Sigma L_j(e_j) + \Sigma f_k M_k(e_k^*) = 0 \tag{2.7}$$

is some homogeneous expression in which L_j and M_k are linear operators not containing the geometric or elastic constants which appear in the static-geometric analogy (they may contain coefficients of the first or second quadratic form of a surface), and e_j and e_k^* are the elements of the first and second groups, respectively, multiplied, perhaps, by dimensionless constants.

On the basis of the static-geometric analogy there exists a homogeneous relation duplicating Expression (2.7):

$$\Sigma L_j(e_j^*) + \Sigma f_k^* M_k(e_k) = 0 \tag{2.8}$$

where f_k^* is a quantity analogous to f_k. Multiplying Equations (2.8) by $i\xi$ and adding the result to Equation (2.7), we obtain

$$\Sigma L_j(e_j + i\xi e_j^*) + \Sigma M_k(f_k e_k^* + i\xi f_k^* e_k) = 0$$

or

$$\Sigma L_j (e_j + i\xi e_j^*) + i\xi \Sigma f_k^* M_k \left(e_k - i\frac{f_k}{\xi f_k^*} e_k^*\right) = 0 \qquad (2.9)$$

It is apparent that the quantities $e_j + i\xi e_j^*$ are quasi-invariants and, in accordance with previous notation, we can write $S_j = e_j + i\xi e_j^*$.

We shall show first of all that

$$e_k - i\frac{f_k}{\xi f_k^*} e_k^*$$

are quasi-invariants. To do so, we note that

$$\left|\frac{f_k}{\xi f_k^*}\right| = |F|, \qquad \frac{f_k}{\xi f_k^*} \longleftarrow\longrightarrow \frac{f_k^*}{-f_k/\xi} = -\frac{1}{f_k/\xi f_k^*}$$

and consequently

$$-\frac{f_k}{\xi^2 f_k^*} = \lambda_k \qquad \left(\lambda_k \longleftarrow\longrightarrow \frac{1}{\lambda_k}\right) \qquad (2.10)$$

Here λ_k is a dimensionless constant satisfying the conditions of the note in Section 1. Thus

$$e_k - i\frac{f_k}{\xi f_k^*} e_k^* = e_k + i\xi\lambda_k e_k^*$$

are quasi-invariants. In accordance with the previous notations we have

$$e_k + i\xi\lambda_k e_k^* = \frac{1}{2}[(1+\lambda_k)S_k + (1-\lambda_k)\overline{S}_k]$$

Consequently, Equation (2.9) can be written in the final form

$$\Sigma L_j(S_j) + \frac{1}{2} i\xi\Sigma f_k^* M_k [(1+\lambda_k)S_k + (1-\lambda_k)\overline{S}_k] = 0 \qquad (2.11)$$

where λ_k is given by (2.10).

Example. For a shallow shell referred to the lines of curvature (α,β) we obtain a first equation in the form

$$\frac{1}{R_2}\frac{\partial^2 c}{\partial\alpha^2} + \frac{1}{R_1}\frac{\partial^2 c}{\partial\beta^2} - \frac{2Eh^3}{3(1-\mu^2)}\Delta\Delta w = 0$$

in which c is the third stress function and w is the displacement normal to the middle surface. This equation can be represented in the form

$$L(c) + fM(w) = 0$$

where

$$L = \frac{1}{R_2}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{R_1}\frac{\partial^2}{\partial\beta^2}, \qquad M = \Delta\Delta, \qquad f = -\frac{2Eh^3}{3(1-\mu^2)}$$

By the static-geometric analogy $f^* = 1/2\, Eh$ and, consequently, from (2.10) we have

$$\lambda = -\left[-\frac{2Eh^3}{3(1-\mu^2)} \Big/ \left(\frac{2Eh^2}{\sqrt{3(1-\mu^2)}}\right)^2 \frac{1}{2Eh}\right] = 1$$

Thus, from (2.10), we find that

$$L(c) + \frac{ih}{\sqrt{3(1-\mu^2)}} M(c) = 0$$

since $c \leftarrow \rightarrow w$, because

$$\frac{1}{R_2}\frac{\partial^2 c}{\partial \alpha^2} + \frac{1}{R_1}\frac{\partial^2 c}{\partial \beta^2} + \frac{ih}{\sqrt{3(1-\mu^2)}} \Delta\Delta c = 0$$

3. Complementary equations in the theory of thin anisotropic shells. The result of the previous section enables us to write down Hooke's equation in the form of three non-differential linear equations, relating the complex moments G_1, G_2, H_1 and the complex forces T_1, T_2 and S_1. We shall call these the complementary equations.

We shall make use of the relations

$$G_1 = -\frac{2h^3}{3} A_{22}\left(\frac{A_{11}}{A_{22}}\varkappa_1 + \frac{A_{12}}{A_{22}}\varkappa_2 + 2\frac{A_{13}}{A_{22}}\tau\right)$$

$$G_2 = -\frac{2h^3}{3} A_{22}\left(\frac{A_{21}}{A_{22}}\varkappa_1 + \varkappa_2 + 2\frac{A_{23}}{A_{22}}\tau\right)$$

$$H_1 = -H_2 = \frac{2h^3}{3} A_{22}\left(\frac{A_{31}}{A_{22}}\varkappa_1 + \frac{A_{32}}{A_{22}}\varkappa_2 + 2\frac{A_{33}}{A_{22}}\tau\right)$$

Employing the method given above, we can now write down the complementary equations in the form

$$\boldsymbol{G}_1 = \frac{ic_3}{2}\{a_{21}[(1+\lambda_1)\boldsymbol{T}_1 + (1-\lambda_1)\overline{\boldsymbol{T}}_1] + a_{22}[(1+\lambda_2)\boldsymbol{T}_2 + (1-\lambda_2)\overline{\boldsymbol{T}}_2] + a_{23}[(1+\lambda_3)\boldsymbol{S}_1 + (1-\lambda_3)\overline{\boldsymbol{S}}_1]\}$$

$$\boldsymbol{G}_2 = \frac{ic_3}{2}\left\{a_{11}\left[\left(1+\frac{1}{\lambda_1}\right)\boldsymbol{T}_1 + \left(1-\frac{1}{\lambda_2}\right)\overline{\boldsymbol{T}}_1\right] + \boldsymbol{a}_{12}[(1+\lambda_4)\boldsymbol{T}_2 + (1-\lambda_4)\overline{\boldsymbol{T}}_2] + a_{13}[(1+\lambda_5)\boldsymbol{S}_1 + (1-\lambda_5)\boldsymbol{S}_1]\right\} \quad (3.1)$$

$$\boldsymbol{H}_1 = \frac{ic_3}{4}\{a_{31}[(1+\lambda_6)\boldsymbol{T}_1 + (1-\lambda_6)\overline{\boldsymbol{T}}_1] + a_{32}[(1+\lambda_7)\boldsymbol{T}_2 + (1-\lambda_7)\overline{\boldsymbol{T}}_2] + a_{33}[(1+\lambda_8)\boldsymbol{S}_1 + (1-\lambda_8)\overline{\boldsymbol{S}}_1]\}$$

where

$$c_3 = h\sqrt[4]{\frac{A_{11}A_{22}}{9a_{11}a_{22}}}, \quad \lambda_1 = \frac{A_{12}}{a_{21}}\sqrt{\frac{a_{11}a_{21}}{A_{11}A_{22}}}, \quad \lambda_2 = \sqrt{\frac{A_{11}a_{11}}{A_{22}a_{22}}}$$

$$\lambda_3 = -\frac{2A_{13}}{a_{13}}\sqrt{\frac{a_{11}a_{22}}{A_{11}A_{22}}}, \quad \lambda_4 = \frac{A_{21}}{a_{12}}\sqrt{\frac{a_{11}a_{22}}{A_{11}A_{22}}}, \quad \lambda_5 = -\frac{2A_{23}}{a_{13}}\sqrt{\frac{a_{11}a_{22}}{A_{11}A_{22}}}$$

$$\lambda_6 = -\frac{2A_{32}}{a_{31}}\sqrt{\frac{a_{11}a_{22}}{A_{11}A_{22}}}, \quad \lambda_7 = -\frac{2A_{31}}{a_{32}}\sqrt{\frac{a_{11}a_{22}}{A_{11}A_{22}}}, \quad \lambda_8 = \frac{4A_{33}}{a_{33}}\sqrt{\frac{a_{11}a_{22}}{A_{11}A_{22}}}$$

Thus, the sets of equations (2.1), (2.4) and (3.1) combine all the basic equations of the theory of thin homogeneous shells.

Appendix. Using the notation given in [1], we can write down the expressions for Hooke's law [3]:

shear forces

$$T_1 = 2hA_{22}\left(\frac{A_{11}}{A_{22}}\varepsilon_1 + \frac{A_{12}}{A_{22}}\varepsilon_2 + \frac{A_{13}}{A_{22}}\omega\right)$$

$$T_2 = 2hA_{22}\left(\frac{A_{21}}{A_{22}}\varepsilon_1 + \varepsilon_2 + \frac{A_{23}}{A_{22}}\omega\right)$$

$$S_1 = -S_2 = 2hA_{22}\left(\frac{A_{31}}{A_{22}}\varepsilon_1 + \frac{A_{32}}{A_{22}}\varepsilon_2 + \frac{A_{33}}{A_{22}}\omega\right)$$

moments

$$G_1 = -\frac{2h^3}{3}A_{22}\left(\frac{A_{11}}{A_{22}}\varkappa_1 + \frac{A_{12}}{A_{22}}\varkappa_2 + 2\frac{A_{13}}{A_{22}}\tau\right)$$

$$G_2 = -\frac{2h^3}{3}A_{22}\left(\frac{A_{21}}{A_{22}}\varkappa_1 + \varkappa_2 + 2\frac{A_{23}}{A_{22}}\tau\right)$$

$$H_1 = -H_2 = \frac{2h^3}{3}A_{22}\left(\frac{A_{31}}{A_{22}}\varkappa_1 + \frac{A_{32}}{A_{22}}\varkappa_2 + 2\frac{A_{33}}{A_{22}}\tau\right)$$

In terms of the strain components, Hooke's law can be written in the form

$$\varepsilon_1 = \frac{a_{11}}{2h}\left(T_1 + \frac{a_{12}}{a_{11}}T_2 + \frac{a_{13}}{a_{11}}S_1\right)$$

$$\varepsilon_2 = \frac{a_{11}}{2h}\left(\frac{a_{21}}{a_{11}}T_1 + \frac{a_{22}}{a_{11}}T_2 + \frac{a_{23}}{a_{11}}S_1\right)$$

$$\omega = \frac{a_{11}}{2h}\left(\frac{a_{31}}{a_{11}}T_1 + \frac{a_{32}}{a_{11}}T_2 + \frac{a_{33}}{a_{11}}S_1\right)$$

$$\varkappa_1 = -\frac{3}{2h^3}a_{11}\left(G_1 + \frac{a_{12}}{a_{11}}G_2 - \frac{a_{13}}{a_{11}}H_1\right), \qquad \varkappa_2 = -\frac{3}{2h^3}a_{11}\left(\frac{a_{21}}{a_{11}}G_1 + \frac{a_{22}}{a_{11}}G_2 - \frac{a_{23}}{a_{11}}H_1\right)$$

$$\tau = -\frac{3}{2h^3}a_{11}\left(\frac{a_{31}}{2a_{11}}G_1 + \frac{a_{32}}{2a_{11}}G_2 - \frac{a_{33}}{2a_{11}}H_1\right)$$

Hooke's law expresses the fact that there is a plane of elastic symmetry, tangential at every point in a three-dimensional medium occupied by the shell, to a surface equidistant from the middle surface [3, 9, 10].

The elastic constants A_{ij} and the strain coefficients a_{ij} are given by the formulas

$$A_{ij} = \frac{(-1)^{i+j}a_{ij}}{D}, \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{matrix} a_{11} = \dfrac{1}{E_\alpha}, & a_{12} = -\dfrac{\mu_\mu}{E_\alpha}, & a_{13} = \dfrac{\eta_\alpha}{E_\alpha} \\ a_{21} = -\dfrac{\mu_\beta}{E_\beta}, & a_{22} = \dfrac{1}{E_\beta}, & a_{23} = \dfrac{\eta_\beta}{E_\beta} \\ a_{31} = \dfrac{\nu_\alpha}{G}, & a_{32} = \dfrac{\nu_\beta}{C}, & a_{33} = \dfrac{1}{G} \end{matrix}$$

where $(-1)^{i+j}a_{ij}$ is an algebraic complement of the element a_{ij}.

The physical meaning of these constants [9] is obvious.

The relations of the static-geometric analogy for the anisotropic case are [3]

$$F_1 = 2hA_{11}, \quad D_1 = \frac{3}{2h^3}a_{11}, \quad F_2 = 2hA_{22}, \quad D_2 = \frac{3}{2h^3}a_{22}, \quad F_1 \leftarrow\rightarrow -D_2, \quad F_2 \leftarrow\rightarrow -D_1$$

$$\frac{A_{12}}{A_{22}} \leftarrow\rightarrow \frac{a_{21}}{a_{11}}, \quad \frac{A_{21}}{A_{22}} \leftarrow\rightarrow \frac{a_{12}}{a_{11}}, \quad 2\frac{A_{13}}{A_{22}} \leftarrow\rightarrow -\frac{a_{23}}{a_{11}}, \quad 2\frac{A_{23}}{A_{22}} \leftarrow\rightarrow -\frac{a_{13}}{a_{11}}$$

$$2\frac{A_{31}}{A_{22}} \leftarrow\rightarrow -\frac{a_{32}}{a_{11}}, \quad 2\frac{A_{32}}{A_{22}} \leftarrow\rightarrow -\frac{a_{31}}{a_{11}}, \quad 4\frac{A_{33}}{A_{22}} \leftarrow\rightarrow \frac{a_{33}}{a_{11}}$$

BIBLIOGRAPHY

1. Goldenveizer, A.L., *Teoriia uprugikh tonkikh obolochek (Theory of Thin Elastic Shells)*. Gostekhizdat, 1953.

2. Visarion, V. and Stanescu, C., Extinderea analogiei statico geometrice pentru invelitorile elastice subtiri cu ortotropie de material. *Comun. Acad. RPR* Vol. 7, No. 3, 1957.

3. Stanescu, C. and Visarion, V., Extinderea analogiei statico geometrice pentru invelitorile elastice subtiri cu anizotropie de material. *Studii si Cercetari mec apl. Acad. RPR* Vol. 10, No. 3, 1959.

4. Novozhilov, V.V., *Teoriia tonkikh obolochek (Theory of Thin Shells)*. Sudpromgiz, 1951.

5. Goldenveizer, A.L., Issledovanie napriazhennogo sostoianiia sfericheskoi obolochki (An investigation of the state of stress in a spherical shell). *PMM* Vol. 8, No. 6, 1944.

6. Goldenveizer, A.L., Uravneniia teorii obolochek v peremeshcheniiakh i funktsiiakh napriazhenii (Equations of the theory of shells in terms of displacements and stress functions). *PMM* Vol. 21, No. 6, 1957.

7. Visarion, V., On a method for solving the problem of thin shells. *Rev. Méc. Appl.* Vol. 2, No. 2, 1957. Bucharest.

8. Stanescu, C. and Visarion, V., Statiko-geometricheskaia analogiia dlia tonkikh uprugikh obolochek s ortotropiei materiala i ee primeneniia k raschetu pologikh obolochek i tsilindricheskikh obolochek kruglogo secheniia (The static-geometric analogy for thin elastic shells of orthotropic material and its application to the analysis of shallow shells and cylindrical shells of circular section). *Rev. Méc. Appl.* Vol. 3, No. 1, 1958. Bucharest.

9. Lekhnitskii, S.G., *Teoriia uprugosti anizotropnogo tela (Theory of Elasticity for an Anisotropic Body)*. Gostekhizdat, 1950.

10. Sokolnikoff, I.L., *Mathematical Theory of Elasticity*. McGraw-Hill, 1956.

11. Chernykh, K.F., O variatsionnom printsipe kompleksnoi teorii obolochek (On the variational principle of the complex theory of shells). *PMM* Vol. 22, No. 2, 1958.

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